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Neumann and Neumann-Rosochatius integrable systems from membranes on $AdS_4 \times S^7$

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Abstract: It is known that large class of classical string solutions in the type IIB $AdS_5 \times S^5$ background is related to the Neumann and Neumann-Rosochatius integrable systems, including spiky strings and giant magnons. It is also interesting if these integrable systems can be associated with some membrane configurations in M-theory. We show here that this is indeed the case by presenting explicitly several types of membrane embedding in $AdS_4 \times S^7$ with the searched properties.

Keywords: M-Theory, AdS-CFT Correspondence.

Contents

1. Introduction

The AdS/CFT correspondence predicts that the string theory on $AdS_5 \times S^5$ should be dual to $\mathcal{N} = 4$ SYM theory in four dimensions [1–3]. The spectrum of the string states and of the operators in SYM should be the same. The recent checks of this conjecture *beyond* the supergravity approximation are connected to the idea to search for string solutions, which in the semiclassical limit are related to the anomalous dimensions of certain gauge invariant operators in the planar limit of SYM [4, 5]. On the field theory side, it was found that the corresponding dilatation operator is connected to the Hamiltonian of integrable Heisenberg spin chain [6]. On the string side, it was established that large set of classical string solutions follow from embeddings , which reduce the solution of the string equations of motion and constraints to the study of the Neumann and Neumann-Rosochatius integrable systems in the presence of conformal gauge constraints [7–9].

In [7] it was shown that solitonic solutions of the classical string action on the type IIB $AdS_5 \times S^5$ background that carry charges of the Cartan subalgebra of the global symmetry group can be classified in terms of periodic solutions of the Neumann dynamical system [10], which is Liouville integrable [11]. A particular string solution was also identified, whose classical energy reproduces *exactly* the one-loop anomalous dimension of a certain set of SYM operators with two independent R-charges.

A general class of rotating closed string solutions in $AdS_5 \times S^5$ was shown to be connected to the Neumann-Rosochatius integrable system [12] in [8].

Let us note that the first multi-spin string solutions were found in [13] and [14], where it was pointed out that the classical energy of the strings admitted a perturbative expansion in powers of the 't Hooft coupling λ, and therefore could be compared with perturbative field theory computations. Actually, the very first two-spin string was found in [15] but its

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importance was not understood at that time. Papers [7] and [8] generalized these multi-spin solutions.

The first perturbative field theory computation of anomalous dimensions of long SYM operators was done in [16], where the operators dual to strings from [13] and [14] were identified and exact one-loop matching was shown.

It was found in [9] that, working in conformal gauge, the spiky strings [17, 18] and giant magnons [19]- [39] can be also accommodated by a version of the Neumann-Rosochatius system. The authors of [9] was able to describe in detail a giant magnon solution with two additional angular momenta and to show that it can be interpreted as a superposition of two magnons moving with the same speed. In addition, they considered the spin chain side and described the corresponding state as that of two bound states in the infinite SU(3) spin chain. The Bethe ansatz wave function for such bound state was also constructed.

It was also shown recently that magnon-like dispersion relations can arise from M-theory [27, 34]. That is why, it is interesting if the Neumann and Neumann-Rosochatius integrable systems can be associated with some M2-brane configurations. In this paper, we prove that this is indeed the case by presenting explicitly several types of membrane embedding in $AdS_4 \times S^7$ with the desired properties.

2. Short review of the string case

Our aim here is to briefly describe part of the results obtained in [7, 8] and [9], concerning the correspondence between different type of string solutions on $AdS_5 \times S^5$ in conformal gauge with the Neumann and Neumann-Rosochatius like integrable systems. Then we show how to generalize these results to the case of diagonal worldsheet gauge.

The action for the bosonic part of the classical closed string moving in the $AdS_5 \times S^5$ background, in conformal gauge, can be written as[1]

$$I = -\frac{\sqrt{\lambda}}{4\pi}\int d\tau d\sigma \left[G^{(AdS_5)}_{mn}(x)\partial_a x^m \partial^a x^n + G^{(S^5)}_{pq}(y)\partial_a y^p \partial^a y^q\right], \qquad \sqrt{\lambda} = 2\pi R^2 T, \tag{2.1}$$

where the two metrics are given by

$$\begin{aligned}\left(ds^2\right)_{AdS_5} &= -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho\left(d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\varphi^2\right), &(2.2)\\ \left(ds^2\right)_{S_5} &= d\gamma^2 + \cos^2\gamma d\varphi_3^2 + \sin^2\gamma\left(d\psi^2 + \cos^2\psi d\varphi_1^2 + \sin^2\psi d\varphi_2^2\right). &(2.3)\end{aligned}$$

The action (2.1) can be represented as action for the $O(6) \times \mathrm{SO}(4,2)$ sigma-model

$$I = \frac{\sqrt{\lambda}}{4\pi}\int d\tau d\sigma \left(L_S + L_{AdS}\right), \tag{2.4}$$

where

$$\begin{aligned} L_S &= -\frac{1}{2}\partial_a X_M \partial^a X_M + \frac{1}{2}\Lambda\left(X_M X_M - 1\right),\\ M &= 1,\dots,6, &(2.5)\\ L_{AdS} &= -\frac{1}{2}\eta_{MN}\partial_a Y_M \partial^a Y_N + \frac{1}{2}\tilde{\Lambda}\left(\eta_{MN}Y_M Y_N + 1\right), &(2.6)\\ M &= 0,\dots,5, \qquad \eta_{MN} = (-1,1,1,1,1,-1).\end{aligned}$$

[1] We follow the notation of [7].

The embedding coordinates X_M, Y_M are related to the ones in (2.2), (2.3) as follows

$$\begin{aligned} X_1 + iX_2 &= \sin\gamma\cos\psi e^{i\varphi_1}, \\ X_3 + iX_4 &= \sin\gamma\sin\psi e^{i\varphi_2}, \\ X_5 + iX_6 &= cos\gamma e^{i\varphi_3}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} Y_1 + iY_2 &= \sinh\rho\sin\theta e^{i\phi}, \\ Y_3 + iY_4 &= \sinh\rho\cos\theta e^{i\varphi}, \\ Y_5 + iY_0 &= \cosh\rho e^{it}. \end{aligned} \tag{2.8}$$

The action (2.4) must be supplemented with the two conformal gauge constraints.

Further on, the following ansatz for the string embedding has been proposed in [7]

$$\begin{aligned} &Y_1, \dots, Y_4 = 0, \qquad\qquad Y_5 + iY_0 = e^{i\kappa\tau}, \\ &X_1 + iX_2 = x_1(\sigma)e^{i\omega_1\tau}, \quad X_3 + iX_4 = x_2(\sigma)e^{i\omega_2\tau}, \quad X_5 + iX_6 = x_3(\sigma)e^{i\omega_3\tau}. \end{aligned} \tag{2.9}$$

It corresponds to string located at the center of AdS_5 and rotating in S^5. Replacing (2.9) into (2.5), (2.6), one obtains the string Lagrangian (prime is used for $d/d\sigma$)

$$L_S + L_{AdS} = -\frac{1}{2}\left[\sum_{i=1}^{3}\left(x_i'^2 - \omega_i^2 x_i^2\right) + \kappa^2\right] + \frac{1}{2}\Lambda\left(\sum_{i=1}^{3} x_i^2 - 1\right).$$

After changing the overall sign and neglecting the constant term as in [7], one arrives at

$$L = \frac{1}{2}\sum_{i=1}^{3}\left(x_i'^2 - \omega_i^2 x_i^2\right) + \frac{1}{2}\Lambda\left(\sum_{i=1}^{3} x_i^2 - 1\right). \tag{2.10}$$

L describes three dimensional harmonic oscillator constrained to remain on a unit two-sphere. This is particular case of the n-dimensional Neumann dynamical system [10], which is Liouville *integrable* [11]. In the case under consideration, the only nontrivial Virasoro constraint implies that the energy H of the Neumann system is given by

$$H = \frac{1}{2}\sum_{i=1}^{3}\left(x_i'^2 + \omega_i^2 x_i^2\right) = \frac{1}{2}\kappa^2. \tag{2.11}$$

In order to obtain the relevant closed string solutions, we should impose periodicity conditions on x_i:

$$x_i(\sigma) = x_i(\sigma + 2\pi).$$

Another string embedding is possible, related to Neumann like integrable system [7]

$$Y_1 + iY_2 = y_1(\sigma)e^{i\omega_1\tau}, \quad Y_3 + iY_4 = y_2(\sigma)e^{i\omega_2\tau}, \quad Y_5 + iY_0 = y_3(\sigma)e^{i\omega_3\tau}. \tag{2.12}$$

It corresponds to multi-spin strings rotating not in S^5 but in AdS_5 instead. Now $t = \omega_3\tau$, so the equality $\omega_3 = \kappa$ holds. The relevant effective mechanical system describing this class of rotating solutions has the following Lagrangian

$$\tilde{L} = \frac{1}{2}\eta_{ij}\left(y_i'y_j' - \omega_i^2 y_i y_j\right) + \frac{1}{2}\tilde{\Lambda}\left(\eta_{ij}y_i y_j - 1\right), \qquad \eta_{ij} = diag(-1,-1,1). \tag{2.13}$$

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Comparing this with the Neumann Lagrangian (2.10), one concludes that (2.13) corresponds to a system, which is similar to the Neumann integrable system, but with indefinite signature - δ_{ij} replaced by η_{ij}. The relation to the S^5 case is through the analytic continuation

$$x_1 \to iy_1, \qquad x_2 \to iy_2.$$

The results presented above have been generalized in [8] to correspondence between closed strings in $AdS_5 \times S^5$ and the Neumann-Rosochatius integrable system [12]. This has been achieved by using more general ansatz for the string embedding. Two such types of embedding have been given in [8]. The first one is[2]

$$\begin{aligned}
Y_1, \dots, Y_4 &= 0, \\
Y_5 + iY_0 &= e^{i\kappa\tau}, \\
X_1 + iX_2 &= r_1(\sigma) e^{i[\omega_1 \tau + \alpha_1(\sigma)]}, \\
X_3 + iX_4 &= r_2(\sigma) e^{i[(\omega_2 \tau + \alpha_2(\sigma)]}, \\
X_5 + iX_6 &= r_3(\sigma) e^{i[\omega_3 \tau + \alpha_3(\sigma)]}.
\end{aligned} \tag{2.14}$$

To find the corresponding closed string solutions, one imposes the periodicity conditions

$$r_i(\sigma + 2\pi) = r_i(\sigma), \qquad \alpha_i(\sigma + 2\pi) = \alpha_i + 2\pi m_i, \qquad m_i = 0, \pm 1, \pm 2, \dots .$$

The ansatz (2.14) leads to the following Lagrangian

$$L = \frac{1}{2} \sum_{i=1}^{3} \left(r_i'^2 + r_i^2 \alpha_i'^2 - \omega_i^2 r_i^2 \right) - \frac{1}{2} \Lambda \left(\sum_{i=1}^{3} r_i^2 - 1 \right). \tag{2.15}$$

The equations of motion for the variables $\alpha_i(\sigma)$ can be easily integrated once

$$\alpha_i' = \frac{v_i}{r_i^2}, \qquad v_i = constants. \tag{2.16}$$

Substituting (2.16) back into (2.15), one receives an effective Lagrangian for the three real coordinates $r_i(\sigma)$[3]

$$L = \frac{1}{2} \sum_{i=1}^{3} \left(r_i'^2 - \omega_i^2 r_i^2 - \frac{v_i^2}{r_i^2} \right) - \frac{1}{2} \Lambda \left(\sum_{i=1}^{3} r_i^2 - 1 \right). \tag{2.17}$$

When α_i are constants, i.e. $v_i = 0$, (2.17) reduces to the Neumann Lagrangian (2.10). For non-zero v_i, the Lagrangian (2.17) describes the Neumann-Rosochatius integrable system. The Virasoro constraints take the form

$$\begin{aligned}
\sum_{i=1}^{3} \left(r_i'^2 + \omega_i^2 r_i^2 + \frac{v_i^2}{r_i^2} \right) &= \kappa^2, \\
\sum_{i=1}^{3} \omega_i v_i &= 0.
\end{aligned}$$

[2] We follow the notation of [8].

[3] Following [8], we change the signs of the terms $\sim \alpha_i'^2$.

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As a consequence of the second equality, only two of the three integrals of motion v_i are independent of ω_i.

The second type of embedding proposed in [8] is for the case when the string rotates in both AdS_5 and S^5. It is given by (2.14) for $X_1, \ldots, X_6$ and

$$\begin{aligned} Y_5 + iY_0 &= \mathrm{r}_0(\sigma)e^{i[w_0\tau+\beta_0(\sigma)]}, \\ Y_1 + iY_2 &= \mathrm{r}_1(\sigma)e^{i[w_1\tau+\beta_1(\sigma)]}, \\ Y_3 + iY_4 &= \mathrm{r}_2(\sigma)e^{i[w_2\tau+\beta_2(\sigma)]}. \end{aligned} \tag{2.18}$$

To satisfy the closed string periodicity conditions, one needs the following equalities to hold (k_r are integers)

$$\mathrm{r}_r(\sigma + 2\pi) = \mathrm{r}_r(\sigma), \quad \beta_r(\sigma + 2\pi) = \beta_r(\sigma) + 2\pi k_r, \quad r = 0, 1, 2.$$

Requiring the time coordinate to be single-valued (considering a universal cover of AdS_5), i.e. ignoring windings in time direction, and also renaming w_0 to κ, one obtains

$$k_0 = 0, \quad w_0 \equiv \kappa.$$

The mechanical system corresponding to the above embedding is described by the sum of the Lagrangian (2.17) and the following one

$$\tilde{L} = \frac{1}{2}\eta^{rs}\left(\mathrm{r}_r'\mathrm{r}_s' - w_r^2\mathrm{r}_s\mathrm{r}_s - \frac{u_r u_s}{\mathrm{r}_r\mathrm{r}_s}\right) - \frac{1}{2}\tilde{\Lambda}\left(\eta^{rs}\mathrm{r}_r\mathrm{r}_s + 1\right), \quad \eta^{rs} = (-1, 1, 1), \tag{2.19}$$

which represents an integrable system too.

For the present case, the equations of motion for r_i and r_s, following from (2.17) and (2.19) respectively, decouple. However, in the conformal gauge constraints, the variables of the two Neumann-Rosochatius systems are mixed. More precisely, the Virasoro constraints now read

$$\begin{aligned} \mathrm{r}_0'^2 + \kappa^2\mathrm{r}_0^2 + \frac{u_0^2}{\mathrm{r}_0^2} &= \sum_{a=1}^{2}\left(\mathrm{r}_a'^2 + w_a^2\mathrm{r}_a^2 + \frac{u_a^2}{\mathrm{r}_a^2}\right) + \sum_{i=1}^{3}\left(r_i'^2 + \omega_i^2 r_i^2 + \frac{v_i^2}{r_i^2}\right), \\ \kappa u_0 &= \sum_{a=1}^{2} w_a u_a + \sum_{i=1}^{3}\omega_i v_i, \end{aligned}$$

where

$$\mathrm{r}_0^2 - \sum_{a=1}^{2}\mathrm{r}_a^2 = 1, \quad \sum_{i=1}^{3} r_i^2 = 1.$$

We also require the periodicity conditions [8]

$$u_s \int_0^{2\pi} \frac{d\sigma}{\mathrm{r}_s^2(\sigma)} = 2\pi k_s$$

to be fulfilled. Then $k_0 = 0$ implies $u_0 = 0$ as a consequence of the single-valuedness of the time coordinate t.

The authors of [9], inspired by the recent development in string/CFT duality, proposed new string embedding, which incorporates the spiky strings [17, 18] and giant magnons [19]–[28] on S^5. They showed that such string solutions can be also accommodated by a version of the Neumann-Rosochatius integrable system. The appropriate embedding is given by

$$\begin{aligned} Y_1, \dots, Y_4 &= 0, \\ Y_5 + iY_0 &= e^{i\kappa\tau}, \\ X_1 + iX_2 &= r_1(\xi) e^{i[\omega_1\tau + \mu_1(\xi)]}, \\ X_3 + iX_4 &= r_2(\xi) e^{i[(\omega_2\tau + \mu_2(\xi)]}, \\ X_5 + iX_6 &= r_3(\xi) e^{i[\omega_3\tau + \mu_3(\xi)]}, \end{aligned} \tag{2.20}$$

where

$$\xi = \alpha\sigma + \beta\tau.$$

This ansatz leads to the Lagrangian [9]

$$L = \sum_{i=1}^{3} \left[\left(\alpha^2 - \beta^2\right) r_i'^2 - \frac{1}{\alpha^2 - \beta^2} \frac{C_i^2}{r_i^2} - \frac{\alpha^2}{\alpha^2 - \beta^2} \omega_i^2 r_i^2 \right] + \Lambda \left(\sum_{i=1}^{3} r_i^2 - 1 \right), \tag{2.21}$$

which describes the standard Neumann-Rosochatius integrable system. The corresponding Hamiltonian is

$$H = \sum_{i=1}^{3} \left[\left(\alpha^2 - \beta^2\right) r_i'^2 + \frac{1}{\alpha^2 - \beta^2} \frac{C_i^2}{r_i^2} + \frac{\alpha^2}{\alpha^2 - \beta^2} \omega_i^2 r_i^2 \right].$$

The Virasoro constraints are satisfied if

$$H = \frac{\alpha^2 + \beta^2}{\alpha^2 - \beta^2} \kappa^2, \qquad \sum_{i=1}^{3} \omega_i C_i + \beta\kappa^2 = 0.$$

The periodicity conditions read

$$r_i(\xi + 2\pi\alpha) = r_i(\xi), \qquad \mu_i(\xi + 2\pi\alpha) = \mu_i(\xi) + 2\pi n_i,$$

where n_i are integer winding numbers. The second condition implies

$$\frac{C_i}{2\pi} \int_0^{2\pi\alpha} \frac{d\xi}{r_i^2} = (\alpha^2 - \beta^2) n_i - \alpha\beta\omega_i.$$

Thus the general solution for the ansatz (2.20) can be constructed in terms of the usual solutions of the Neumann-Rosochatius system. There are five independent integrals of motion, which reduce the equations of motion to a system of first order differential equations that can be directly integrated [7].

All the above results are obtained in conformal gauge. In order to make connection with the membrane case, we will formulate the problem in the framework of the more general diagonal worldsheet gauge. In this gauge, the Polyakov action and constraints are given by

$$S_S = \int d^2\xi \mathcal{L}_S = \int d^2\xi \frac{1}{4\lambda^0}\Big[G_{00} - \left(2\lambda^0 T\right)^2 G_{11}\Big], \tag{2.22}$$

$$G_{00} + \left(2\lambda^0 T\right)^2 G_{11} = 0, \tag{2.23}$$

$$G_{01} = 0, \tag{2.24}$$

where

$$G_{mn} = g_{MN}\partial_m X^M \partial_n X^N,$$
$$\left[\partial_m = \partial/\partial\xi^m, \quad m = (0,1), \quad (\xi^0,\xi^1) = (\tau,\sigma), \quad M = (0,1,\ldots,9)\right],$$

is the induced metric and λ^0 is Lagrange multiplier. The usually used conformal gauge corresponds to $2\lambda^0 T = 1$.

The general string embedding in $AdS_5 \times S^5$ of the type we are interested in can be written as

$$Z_s = R\mathrm{r}_s(\xi^m)e^{i\phi_s(\xi^m)}, \quad s = (0,1,2), \quad \eta^{rs}\mathrm{r}_r\mathrm{r}_s + 1 = 0, \quad \eta^{rs} = (-1,1,1),$$
$$W_i = Rr_i(\xi^m)e^{i\varphi_i(\xi^m)}, \quad i = (1,2,3), \quad \delta_{ij}r_ir_j - 1 = 0. \tag{2.25}$$

For this embedding, the induced metric takes the form

$$G_{mn} = \eta^{rs}\partial_{(m}Z_r\partial_{n)}\bar{Z}_s + \delta_{ij}\partial_{(m}W_i\partial_{n)}\bar{W}_j = \tag{2.26}$$
$$R^2\left[\sum_{r,s=0}^{2}\eta^{rs}\left(\partial_m\mathrm{r}_r\partial_n\mathrm{r}_s + \mathrm{r}_r^2\partial_m\phi_r\partial_n\phi_s\right) + \sum_{i=1}^{3}\left(\partial_m r_i\partial_n r_i + r_i^2\partial_m\varphi_i\partial_n\varphi_i\right)\right].$$

The expression (2.26) for G_{mn} must be replaced into (2.22), (2.23) and (2.24). Correspondingly, the string Lagrangian will be

$$\mathcal{L} = \mathcal{L}_S + \Lambda_A(\eta^{rs}\mathrm{r}_r\mathrm{r}_s + 1) + \Lambda_S(\delta_{ij}r_ir_j - 1), \tag{2.27}$$

where Λ_A and Λ_S are Lagrange multipliers.

As an example, let us choose the following ansatz for the string embedding of the type (2.25)

$$Z_0 = Re^{i\kappa\tau}, \quad Z_1 = Z_2 = 0, \quad W_i = Rr_i(\sigma)e^{i\omega_i\tau},$$

which implies

$$\mathrm{r}_0 = 1, \quad \mathrm{r}_1 = \mathrm{r}_2 = 0; \quad \phi_0 = \kappa\tau, \quad \varphi_i = \omega_i\tau.$$

Then (2.27) reduces to (prime is used for $d/d\sigma$)

$$\mathcal{L} = -\frac{R^2}{4\lambda^0}\left\{\sum_{i=1}^{3}\left[(2\lambda^0 T)^2 r_i'^2 - \omega_i^2 r_i^2\right] + \kappa^2\right\} + \Lambda_S\left(\sum_{i=1}^{3} r_i^2 - 1\right).$$

After changing the overall sign and neglecting the constant term as in [7], one obtains

$$L = \frac{R^2}{4\lambda^0}\sum_{i=1}^{3}\left[(2\lambda^0 T)^2 r_i'^2 - \omega_i^2 r_i^2\right] + \Lambda_S\left(\sum_{i=1}^{3} r_i^2 - 1\right),$$

which in conformal gauge ($2\lambda^0 T = 1$) is equivalent to (2.10). The constraint (2.23) gives the corresponding Hamiltonian

$$H \sim \sum_{i=1}^{3}\left[(2\lambda^0 T)^2 r_i'^2 + \omega_i^2 r_i^2\right] = \kappa^2.$$

The other constraint (2.24) is satisfied identically.

In the same way, one can generalize the other previously obtained results [7 − 9] to the case of diagonal worldsheet gauge.

3. Membranes on $AdS_4 \times S^7$

Turning to the membrane case, let us first write down the gauge fixed membrane action and constraints in diagonal worldvolume gauge, we are going to work with:

$$S_M = \int d^3\xi \mathcal{L}_M = \int d^3\xi \left\{\frac{1}{4\lambda^0}\left[G_{00} - \left(2\lambda^0 T_2\right)^2 \det G_{ij}\right] + T_2 C_{012}\right\}, \tag{3.1}$$

$$G_{00} + \left(2\lambda^0 T_2\right)^2 \det G_{ij} = 0, \tag{3.2}$$

$$G_{0i} = 0. \tag{3.3}$$

They *coincide* with the frequently used gauge fixed Polyakov type action and constraints after the identification $2\lambda^0 T_2 = L = const$, where λ^0 is Lagrange multiplier and T_2 is the membrane tension. In (3.1)–(3.3), the fields induced on the membrane worldvolume G_{mn} and C_{012} are given by

$$\begin{aligned} G_{mn} &= g_{MN}\partial_m X^M \partial_n X^N, \\ C_{012} &= c_{MNP}\partial_0 X^M \partial_1 X^N \partial_2 X^P, \end{aligned} \tag{3.4}$$

$$\begin{aligned} \partial_m &= \partial/\partial\xi^m, \\ m &= (0, i) = (0, 1, 2), \end{aligned} \tag{3.5}$$

$$\begin{aligned} (\xi^0, \xi^1, \xi^2) &= (\tau, \sigma_1, \sigma_2), \\ M &= (0, 1, \ldots, 10), \end{aligned} \tag{3.6}$$

where g_{MN} and c_{MNP} are the components of the target space metric and 3-form gauge field respectively.

Searching for membrane configurations in $AdS_4 \times S^7$, which correspond to the Neumann or Neumann-Rosochatius integrable systems, we should first eliminate the membrane interaction with the background 3-form field on AdS_4, to ensure more close analogy with the strings on $AdS_5 \times S^5$. To make our choice, let us write down the background. It can be parameterized as follows

$$\begin{aligned} ds^2 &= (2l_p\mathcal{R})^2 \left[-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho \left(d\alpha^2 + \sin^2\alpha d\beta^2\right) + 4d\Omega_7^2\right], \\ c_{(3)} &= (2l_p\mathcal{R})^3 \sinh^3\rho \sin\alpha dt \wedge d\alpha \wedge d\beta. \end{aligned}$$

Since we want the membrane to have nonzero conserved energy and spin on AdS, the possible choice, for which the interaction with the $c_{(3)}$ field disappears, is to fix the angle α[4]:

$$\alpha = \alpha_0 = const.$$

The metric of the corresponding subspace of AdS_4 is

$$\begin{aligned} ds_{sub}^2 &= (2l_p\mathcal{R})^2 \left(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho \sin^2\alpha_0 d\beta^2\right) \\ &= (2l_p\mathcal{R})^2 \left[-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d(\beta\sin\alpha_0)^2\right]. \end{aligned} \tag{3.7}$$

Therefore, the appropriate membrane embedding into (3.7) and S^7 is

$$\begin{aligned} Z_\mu &= 2l_p\mathcal{R}\mathrm{r}_\mu(\xi^m)e^{i\phi_\mu(\xi^m)}, \qquad \mu=(0,1), \qquad \phi_\mu = (\phi_0,\phi_1) = (t, \beta\sin\alpha_0), \\ &\qquad \eta^{\mu\nu}\mathrm{r}_\mu\mathrm{r}_\nu + 1 = 0, \qquad\qquad \eta^{\mu\nu} = (-1,1), \\ W_a &= 4l_p\mathcal{R}r_a(\xi^m)e^{i\varphi_a(\xi^m)}, \qquad a=(1,2,3,4), \qquad \delta_{ab}r_ar_b - 1 = 0. \end{aligned} \tag{3.8}$$

For this embedding, the induced metric is given by

$$\begin{aligned} G_{mn} &= \eta^{\mu\nu}\partial_{(m}Z_\mu\partial_{n)}\bar{Z}_\nu + \delta_{ab}\partial_{(m}W_a\partial_{n)}\bar{W}_b \\ &= (2l_p\mathcal{R})^2 \left[\sum_{\mu,\nu=0}^{1}\eta^{\mu\nu}\left(\partial_m\mathrm{r}_\mu\partial_n\mathrm{r}_\nu + \mathrm{r}_\mu^2\partial_m\phi_\mu\partial_n\phi_\nu\right) + 4\sum_{a=1}^{4}\left(\partial_m r_a\partial_n r_a + r_a^2\partial_m\varphi_a\partial_n\varphi_a\right)\right]. \end{aligned} \tag{3.9}$$

We will use the expression (3.9) for G_{mn} in (3.1), (3.2) and (3.3). Correspondingly, the membrane Lagrangian becomes

$$\mathcal{L} = \mathcal{L}_M + \Lambda_A(\eta^{\mu\nu}\mathrm{r}_\mu\mathrm{r}_\nu + 1) + \Lambda_S(\delta_{ab}r_ar_b - 1). \tag{3.10}$$

3.1 Membranes and the Neumann system

Here, we propose two membrane embeddings in $AdS_4 \times S^7$ related to the Neumann integrable system.

Let us begin with the following ansatz for the membrane embedding of the type (3.8)

$$Z_0 = 2l_p\mathcal{R}e^{i\kappa\tau}, \qquad Z_1 = 0, \qquad W_a = 4l_p\mathcal{R}r_a(\tau)e^{i\omega_{ai}\sigma_i}. \tag{3.11}$$

This implies

$$\mathrm{r}_0 = 1, \qquad \mathrm{r}_1 = 0, \qquad \phi_0 = \kappa\tau, \qquad \varphi_a = \omega_{ai}\sigma_i.$$

[4] Of course, we can fix the angle β instead of α. Then, in the corresponding subspace of AdS_4, α will be the isometry coordinate associated with the conserved spin. The difference is that β is the isometry coordinate in the initial AdS_4 space.

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Then (3.10) takes the form (over-dot is used for $d/d\tau$)

$$\mathcal{L} = \frac{(4l_p\mathcal{R})^2}{4\lambda^0}\left[\sum_{a=1}^{4}\dot{r}_a^2 - \left(8\lambda^0 T_2 l_p\mathcal{R}\right)^2 \sum_{a<b=1}^{4}(\omega_{a1}\omega_{b2}-\omega_{a2}\omega_{b1})^2 r_a^2 r_b^2 - (\kappa/2)^2\right] \tag{3.12}$$
$$+\Lambda_S\left(\sum_{a=1}^{4} r_a^2 - 1\right).$$

It is clear that for arbitrary and different values of the winding numbers ω_{ai}, the potential terms in the above Lagrangian are of forth order with respect to r_a. As far as we are interested in obtaining membrane configurations with quadratic effective potential, our proposal is to make the following choice ($a, b, c \neq 0$ are constants)

$$\omega_{12} = \omega_{22} = \omega_{31} = \omega_{41} = 0,$$
$$\omega_{32} = \pm\omega_{42} = \omega, \tag{3.13}$$
$$r_3(\tau) = a\sin(b\tau + c),$$
$$r_4(\tau) = a\cos(b\tau + c), \qquad a < 1. \tag{3.14}$$

This reduces the membrane Lagrangian to

$$\mathcal{L} = \frac{(4l_p\mathcal{R})^2}{4\lambda^0}\left[\sum_{a=1}^{2}\dot{r}_a^2 - \left(8\lambda^0 T_2 l_p\mathcal{R}a\omega\right)^2 \sum_{a=1}^{2}\omega_{a1}^2 r_a^2 + (ab)^2 - (\kappa/2)^2\right] + \Lambda_S\left(\sum_{a=1}^{2} r_a^2 + a^2 - 1\right). \tag{3.15}$$

After neglecting the constat terms here, one arrives at

$$L = \frac{(4l_p\mathcal{R})^2}{4\lambda^0}\sum_{a=1}^{2}\left[\dot{r}_a^2 - \left(8\lambda^0 T_2 l_p\mathcal{R}a\omega\right)^2 \omega_{a1}^2 r_a^2\right] + \Lambda_S\left[\sum_{a=1}^{2} r_a^2 - (1-a^2)\right].$$

The Lagrangian L describes two-dimensional harmonic oscillator, constrained to remain on a circle of radius $\sqrt{1-a^2}$. Obviously, this is particular case of the Neumann integrable system. The constraint (3.2) gives the Hamiltonian corresponding to L

$$H \sim \sum_{a=1}^{2}\left[\dot{r}_a^2 + \left(8\lambda^0 T_2 l_p\mathcal{R}a\omega\right)^2 \omega_{a1}^2 r_a^2\right] = (\kappa/2)^2 - (ab)^2,$$

while the remaining constraints (3.3) are satisfied identically.

The next ansatz for membrane embedding we will consider is

$$Z_0 = 2l_p\mathcal{R}e^{i\kappa\tau}, \qquad Z_1 = 0, \qquad W_a = 4l_p\mathcal{R}r_a(\sigma_i)e^{i\omega_a\tau}, \tag{3.16}$$

for which (3.10) reduces to

$$\mathcal{L} = -\frac{(4l_p\mathcal{R})^2}{4\lambda^0}\left[\left(8\lambda^0 T_2 l_p\mathcal{R}\right)^2 \sum_{a<b=1}^{4}(\partial_1 r_a\partial_2 r_b - \partial_2 r_a\partial_1 r_b)^2 - \sum_{a=1}^{4}\omega_a^2 r_a^2 + (\kappa/2)^2\right]$$
$$+\Lambda_S\left(\sum_{a=1}^{4} r_a^2 - 1\right). \tag{3.17}$$

Here we have quadratic potential, but in the general case, the kinetic term is not of the type we are searching for. To fix the problem, we set

$$\begin{aligned} r_1 &= r_1(\sigma_1), \quad r_2 = r_2(\sigma_1), \quad \omega_3 = \pm\omega_4 = \omega, \\ r_3(\sigma_2) &= a\sin(b\sigma_2 + c), \qquad r_4(\sigma_2) = a\cos(b\sigma_2 + c), \quad a < 1. \end{aligned} \tag{3.18}$$

This leads to the Lagrangian (prime is used for $d/d\sigma_1$)[5]

$$L = \frac{(4l_p\mathcal{R})^2}{4\lambda^0}\sum_{a=1}^{2}\left[\left(8\lambda^0 T_2 l_p\mathcal{R}ab\right)^2 r_a'^2 - \omega_a^2 r_a^2\right] + \Lambda_S\left[\sum_{a=1}^{2} r_a^2 - (1-a^2)\right], \tag{3.19}$$

which is already of the Neumann type. The corresponding Hamiltonian is given by the constraint (3.2)

$$H \sim \sum_{a=1}^{2}\left[\left(8\lambda^0 T_2 l_p\mathcal{R}ab\right)^2 r_a'^2 + \omega_a^2 r_a^2\right] = (\kappa/2)^2 - (a\omega)^2.$$

The other two constraints (3.3) are satisfied identically.

3.2 Membranes and the Neumann-Rosochatius system

In this subsection, we propose three different membrane embeddings in $AdS_4 \times S^7$ of the type (3.8), which are connected with particular cases of the Neumann-Rosochatius integrable system.

The first one is

$$Z_0 = 2l_p\mathcal{R}e^{i\kappa\tau}, \quad Z_1 = 0, \quad W_a = 4l_p\mathcal{R}r_a(\tau)e^{i[\omega_{ai}\sigma_i + \alpha_a(\tau)]}. \tag{3.20}$$

It leads to the following membrane Lagrangian

$$\begin{aligned} \mathcal{L} = &\frac{(4l_p\mathcal{R})^2}{4\lambda^0}\left[\sum_{a=1}^{4}\left(\dot{r}_a^2 + r_a^2\dot{\alpha}_a^2\right) - \left(8\lambda^0 T_2 l_p\mathcal{R}\right)^2\sum_{a<b=1}^{4}(\omega_{a1}\omega_{b2} - \omega_{a2}\omega_{b1})^2 r_a^2 r_b^2 - (\kappa/2)^2\right] \\ &+ \Lambda_S\left(\sum_{a=1}^{4} r_a^2 - 1\right). \end{aligned} \tag{3.21}$$

The equations of motion for the variables $\alpha_a(\tau)$ can be easily integrated once and the result is

$$\dot{\alpha}_a(\tau) = \frac{C_a}{r_a^2(\tau)}, \tag{3.22}$$

where C_a are arbitrary integration constants. Substituting (3.22) back into (3.21), one receives an effective Lagrangian for the four real coordinates $r_a(\tau)$[6]

$$\begin{aligned} \mathcal{L} = &\frac{(4l_p\mathcal{R})^2}{4\lambda^0}\left[\sum_{a=1}^{4}\left(\dot{r}_a^2 - \frac{C_a^2}{r_a^2}\right) - \left(8\lambda^0 T_2 l_p\mathcal{R}\right)^2\sum_{a<b=1}^{4}(\omega_{a1}\omega_{b2} - \omega_{a2}\omega_{b1})^2 r_a^2 r_b^2 - (\kappa/2)^2\right] \\ &+ \Lambda_S\left(\sum_{a=1}^{4} r_a^2 - 1\right). \end{aligned} \tag{3.23}$$

[5]After changing the overall sign and neglecting the constant terms.

[6]Following [8], we change the signs of the terms $\sim \dot{\alpha}_a^2$.

To get potential terms $\sim r_a^2$ instead of $\sim r_a^2 r_b^2$, we use once again the choice (3.13). In addition, we put $C_3 = C_4 = 0$. All this reduces the membrane Lagrangian to (after neglecting the constant terms)

$$L = \frac{(4l_p\mathcal{R})^2}{4\lambda^0}\sum_{a=1}^{2}\left[\dot{r}_a^2 - \left(8\lambda^0 T_2 l_p \mathcal{R} a\omega\right)^2 \omega_{a1}^2 r_a^2 - \frac{C_a^2}{r_a^2}\right] + \Lambda_S\left[\sum_{a=1}^{2} r_a^2 - (1-a^2)\right], \quad (3.24)$$

which describes Neumann-Rosochatius type integrable system. For $C_a = 0$, (3.24) reduces to Neumann type Lagrangian. Let us also write down the constraints (3.2), (3.3) for the present case. Actually, the third constraint $G_{02} = 0$ is satisfied identically. The other two read

$$H \sim \sum_{a=1}^{2}\left[\dot{r}_a^2 + \left(8\lambda^0 T_2 l_p \mathcal{R} a\omega\right)^2 \omega_{a1}^2 r_a^2 + \frac{C_a^2}{r_a^2}\right] = (\kappa/2)^2 - (ab)^2,$$

$$\sum_{a=1}^{2}\omega_{a1} C_a = 0.$$

Our proposal for the next type of membrane embedding is

$$Z_0 = 2l_p\mathcal{R}e^{i\kappa\tau}, \quad Z_1 = 0, \quad W_a = 4l_p\mathcal{R} r_a(\sigma_i) e^{i[\omega_a\tau + \alpha_a(\sigma_i)]}, \quad (3.25)$$

for which the Lagrangian (3.10) reduces to

$$\begin{aligned}\mathcal{L} = -\frac{(4l_p\mathcal{R})^2}{4\lambda^0}\Bigg\{ &\left(8\lambda^0 T_2 l_p\mathcal{R}\right)^2 \sum_{a<b=1}^{4}\left[(\partial_1 r_a\partial_2 r_b - \partial_2 r_a \partial_1 r_b)^2\right. \quad (3.26)\\ &+ (\partial_1 r_a \partial_2\alpha_b - \partial_2 r_a\partial_1\alpha_b)^2 r_b^2 + (\partial_1\alpha_a\partial_2 r_b - \partial_2\alpha_a\partial_1 r_b)^2 r_a^2\\ &\left.+ (\partial_1\alpha_a\partial_2\alpha_b - \partial_2\alpha_a\partial_1\alpha_b)^2 r_a^2 r_b^2\right]\\ &+ \sum_{a=1}^{4}\left[\left(8\lambda^0 T_2 l_p\mathcal{R}\right)^2(\partial_1 r_a\partial_2\alpha_a - \partial_2 r_a\partial_1\alpha_a)^2 - \omega_a^2\right] r_a^2 + (\kappa/2)^2\Bigg\}\\ +\Lambda_S &\left(\sum_{a=1}^{4} r_a^2 - 1\right).\end{aligned}$$

If we restrict ourselves to the case (3.18) and

$$\alpha_1 = \alpha_1(\sigma_1), \quad \alpha_2 = \alpha_2(\sigma_1), \quad \alpha_3, \alpha_4 = constants,$$

we obtain

$$\mathcal{L} = -\frac{(4l_p\mathcal{R})^2}{4\lambda^0}\left[\left(8\lambda^0 T_2 l_p\mathcal{R}ab\right)^2\sum_{a=1}^{2}\left(r_a'^2 + r_a^2\alpha_a'^2\right) - \sum_{a=1}^{2}\omega_a^2 r_a^2 + (\kappa/2)^2 - (a\omega)^2\right]. \quad (3.27)$$

After integrating the equations of motion for α_a once and replacing the solution into (3.27), one arrives at[7]

$$L = \frac{(4l_p\mathcal{R})^2}{4\lambda^0}\sum_{a=1}^{2}\left[\left(8\lambda^0 T_2 l_p\mathcal{R}ab\right)^2 r_a'^2 - \omega_a^2 r_a^2 - \left(8\lambda^0 T_2 l_p\mathcal{R}ab\right)^2\frac{C_a^2}{r_a^2}\right] + \Lambda_S\left[\sum_{a=1}^{2} r_a^2 - (1-a^2)\right]. \quad (3.28)$$

[7]After changing the corresponding signs and ignoring the constant terms as before.

The above Lagrangian represents particular case of the Neumann-Rosochatius integrable system. For $C_a = 0$, (3.28) coincides with (3.19). The constraints (3.2), (3.3) for the case under consideration are given by

$$H \sim \sum_{a=1}^{2} \left[\left(8\lambda^0 T_2 l_p \mathcal{R} ab\right)^2 r_a'^2 + \omega_a^2 r_a^2 + \left(8\lambda^0 T_2 l_p \mathcal{R} ab\right)^2 \frac{C_a^2}{r_a^2} \right] = (\kappa/2)^2 - (a\omega)^2,$$

$$\sum_{a=1}^{2} \omega_a C_a = 0, \qquad G_{02} \equiv 0.$$

Our last example of membrane embedding is connected to the spiky strings [17, 18] and giant magnons [19] configurations on S^5. It reads

$$\begin{aligned} Z_0 &= 2l_p \mathcal{R} e^{i\kappa\tau}, \qquad Z_1 = 0, \qquad W_a = 4l_p \mathcal{R} r_a(\xi,\eta) e^{i[\omega_a \tau + \mu_a(\xi,\eta)]}, \\ \xi &= \alpha\sigma_1 + \beta\tau, \qquad \eta = \gamma\sigma_2 + \delta\tau, \end{aligned} \tag{3.29}$$

where α, β, γ, δ are constants. For this ansatz, the membrane Lagrangian (3.10) takes the form ($\partial_\xi = \partial/\partial\xi$, $\partial_\eta = \partial/\partial\eta$)

$$\begin{aligned} \mathcal{L} = -\frac{(4l_p\mathcal{R})^2}{4\lambda^0} \Bigg\{ & \left(8\lambda^0 T_2 l_p \mathcal{R}\alpha\gamma\right)^2 \sum_{a<b=1}^{4} \left[(\partial_\xi r_a \partial_\eta r_b - \partial_\eta r_a \partial_\xi r_b)^2 \right. \\ & + (\partial_\xi r_a \partial_\eta \mu_b - \partial_\eta r_a \partial_\xi \mu_b)^2 r_b^2 + (\partial_\xi \mu_a \partial_\eta r_b - \partial_\eta \mu_a \partial_\xi r_b)^2 r_a^2 \\ & \left. + (\partial_\xi \mu_a \partial_\eta \mu_b - \partial_\eta \mu_a \partial_\xi \mu_b)^2 r_a^2 r_b^2 \right] \\ & + \sum_{a=1}^{4} \left[\left(8\lambda^0 T_2 l_p \mathcal{R}\alpha\gamma\right)^2 (\partial_\xi r_a \partial_\eta \mu_a - \partial_\eta r_a \partial_\xi \mu_a)^2 - \omega_a^2 \right] r_a^2 + (\kappa/2)^2 \Bigg\} \\ + \Lambda_S \left(\sum_{a=1}^{4} r_a^2 - 1 \right). & \end{aligned} \tag{3.30}$$

Now, we choose to consider the particular case

$$\begin{aligned} r_1 &= r_1(\xi), \qquad r_2 = r_2(\xi), \qquad \omega_3 = \pm\omega_4 = \omega, \\ r_3 &= r_3(\eta) = a\sin(b\eta + c), \qquad r_4 = r_4(\eta) = a\cos(b\eta + c), \qquad a < 1, \\ \mu_1 &= \mu_1(\xi), \qquad \mu_2 = \mu_2(\xi), \qquad \mu_3, \mu_4 = constants, \end{aligned}$$

and receive (prime is used for $d/d\xi$)

$$\begin{aligned} \mathcal{L} = -\frac{(4l_p\mathcal{R})^2}{4\lambda^0} \Bigg\{ & \sum_{a=1}^{2} \left[(A^2 - \beta^2) r_a'^2 + (A^2 - \beta^2) r_a^2 \left(\mu_a' - \frac{\beta\omega_a}{A^2 - \beta^2} \right)^2 - \frac{A^2}{A^2 - \beta^2} \omega_a^2 r_a^2 \right] \\ & + (\kappa/2)^2 - a^2(\omega^2 + b^2\delta^2) \Bigg\} + \Lambda_S \left[\sum_{a=1}^{2} r_a^2 - (1 - a^2) \right], \end{aligned} \tag{3.31}$$

where

$$A^2 \equiv \left(8\lambda^0 T_2 l_p \mathcal{R} ab\alpha\gamma\right)^2.$$

A single time integration of the equations of motion for μ_a following from the above Lagrangian gives

$$\mu_a' = \frac{1}{A^2-\beta^2}\left(\frac{C_a}{r_a^2}+\beta\omega_a\right). \tag{3.32}$$

Substituting (3.32) back into (3.31), one obtains the following effective Lagrangian for the coordinates $r_a(\xi)$[8]

$$L = \frac{(4l_p\mathcal{R})^2}{4\lambda^0}\sum_{a=1}^{2}\left[(A^2-\beta^2)r_a'^2 - \frac{1}{A^2-\beta^2}\frac{C_a^2}{r_a^2} - \frac{A^2}{A^2-\beta^2}\omega_a^2 r_a^2\right] + \Lambda_S\left[\sum_{a=1}^{2} r_a^2 - (1-a^2)\right]. \tag{3.33}$$

Let us write down the constraints (3.2), (3.3) for the present case. To achieve more close correspondence with the string on $AdS_5\times S^5$, we want the third one to be satisfied identically. To this end, since $G_{02}\sim (ab)^2\gamma\delta$, we set $\delta=0$, i.e. $\eta=\gamma\sigma_2$. Then, the first two constraints give

$$H\sim\sum_{a=1}^{2}\left[(A^2-\beta^2)r_a'^2+\frac{1}{A^2-\beta^2}\frac{C_a^2}{r_a^2}+\frac{A^2}{A^2-\beta^2}\omega_a^2r_a^2\right] = \frac{A^2+\beta^2}{A^2-\beta^2}\left[(\kappa/2)^2-(a\omega)^2\right],$$

$$\sum_{a=1}^{2}\omega_a C_a + \beta\left[(\kappa/2)^2-(a\omega)^2\right] = 0.$$

The Lagrangian (3.33), in full analogy with the string considerations (see (2.21) above or (2.26) of [9]), corresponds to particular case of the n-dimensional Neumann-Rosochatius integrable system.

3.3 Energy and angular momenta

Here, we will evaluate the explicit expressions for the conserved charges, and will obtain relations between them, for the five membrane configurations considered above.

The energy E and the angular momenta J_a can be computed by using the equalities

$$E=-\int d^2\sigma\frac{\partial\mathcal{L}}{\partial\kappa},\qquad J_a=\int d^2\sigma\frac{\partial\mathcal{L}}{\partial(\partial_0\varphi_a)}.$$

Then, for all ansatzes we used, the energy is given by

$$E = 2^3(\pi l_p\mathcal{R})^2\frac{\kappa}{\lambda^0}. \tag{3.34}$$

For the first embedding (3.11), $J_a=0$ for $a=1,2,3,4$, so the only nontrivial conserved quantity is the membrane energy.

For the second embedding (3.16), one obtains

$$J_a = 2^3(l_p\mathcal{R})^2\frac{\omega_a}{\lambda^0}\int d^2\sigma r_a^2,\qquad a=1,2,3,4.$$

[8]Following [9], we change the overall sign, the signs of the terms $\sim C_a^2$, and discard the constant terms.

We will consider the cases $a = 1, 2$ and $a = 3, 4$ separately. According to (3.18), $r_{1,2} = r_{1,2}(\sigma_1)$, which leads to

$$J_a = \pi(4l_p\mathcal{R})^2\frac{\omega_a}{\lambda^0}\int d\sigma_1 r_a^2(\sigma_1), \qquad a = 1, 2.$$

Combining these two equalities with (3.34) and taking into account the constraint

$$\sum_{a=1}^{2} r_a^2 - (1-a^2) = 0,$$

one arrives at the energy-charge relation

$$\frac{E}{\kappa} = \frac{1}{4(1-a^2)}\left(\frac{J_1}{\omega_1} + \frac{J_2}{\omega_2}\right).$$

As usual, we have linear dependence $E(J_1, J_2)$ before taking the semiclassical limit. We comment on this limit in the next section.

Now, let us turn to the case $a = 3, 4$. In accordance with (3.18), we have

$$J_3 = \pi(4l_p\mathcal{R})^2\frac{\omega a^2}{\lambda^0}\int_0^{2\pi} d\sigma_2 \sin^2(b\sigma_2 + c),$$
$$J_4 = \pm\pi(4l_p\mathcal{R})^2\frac{\omega a^2}{\lambda^0}\int_0^{2\pi} d\sigma_2 \cos^2(b\sigma_2 + c).$$

By using the periodicity conditions

$$r_a(\sigma_i) = r_a(\sigma_i + 2\pi),$$

which imply $b = \pm 1, \pm 2, \ldots$, one obtains

$$J_3 = \pm J_4 = (4\pi l_p\mathcal{R})^2\frac{\omega a^2}{\lambda^0}.$$

In order to reproduce the string case, we can set $\omega = 0$, and thus $J_3 = J_4 = 0$.

For the third embedding (3.20), the angular momenta are given by

$$J_a = 2^5(\pi l_p\mathcal{R})^2\frac{C_a}{\lambda^0}, \qquad a = 1, 2; \qquad J_3 = J_4 = 0.$$

This leads to the energy-charge relation

$$\frac{E}{\kappa} = \frac{1}{8}\left(\frac{J_1}{C_1} + \frac{J_2}{C_2}\right).$$

For the forth embedding (3.25), the expressions for the conserved charges and the relation between them are the same as for the second membrane embedding (3.16).

Finally, for the fifth embedding (3.29), $J_3 = J_4 = 0$ for $\omega = 0$. The other two angular momenta are

$$J_a = \frac{\pi(4l_p\mathcal{R})^2}{\lambda^0\alpha(A^2-\beta^2)}\int d\xi\left(\beta C_a + A^2\omega_a r_a^2\right), \qquad a = 1, 2.$$

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Rewriting (3.34) as[9]

$$E = \frac{4\pi(l_p\mathcal{R})^2\kappa}{\lambda^0\alpha}\int d\xi,$$

we obtain the energy-charge relation

$$\frac{4}{A^2-\beta^2}\left[A^2(1-a^2)+\beta\sum_{a=1}^{2}\frac{C_a}{\omega_a}\right]\frac{E}{\kappa} = \sum_{a=1}^{2}\frac{J_a}{\omega_a},$$

in full analogy with the string case. Namely, for strings on $AdS_5 \times S^5$, the result in conformal gauge is [9]

$$\frac{1}{\alpha^2-\beta^2}\left(\alpha^2+\beta\sum_a\frac{C_a}{\omega_a}\right)\frac{E}{\kappa} = \sum_a\frac{J_a}{\omega_a}.$$

Concluding this section, let us make two remarks.

It may seems that the membrane configurations considered here are chosen randomly. However, they correspond exactly to *all* string embeddings in the $R \times S^5$ subspace of $AdS_5 \times S^5$ solution of type IIB string theory, which are known to lead the Neumann and Neumann-Rosochatius dynamical systems [7−9].

Let us also note that our starting ansatz is a particular case of (2.9) in [40].

4. Concluding remarks

We have found here several types of membrane embedding into the $AdS_4 \times S^7$ background, which are related to the Neumann and Neumann-Rosochatius integrable systems, thus reproducing from M-theory viewpoint part of the results established for strings on $AdS_5 \times S^5$. In particular, our Lagrangian (3.33), being completely analogous to the one given in (2.26) of [9], should lead to the same energy-charge relation for the giant magnon solution with two angular momenta (see also [27, 34]). Moreover, the single spike solutions of [41] can be reproduced also from membranes on $AdS_4 \times S^7$ [42]. In addition, one can consider the correspondence between the Neumann and Neumann-Rosochatius integrable systems arising from membranes and the continuous limit of integrable spin chains at the level of actions, as is done in [43].

Besides, it is interesting to clarify the relationship with the constant radii solutions for membranes on $AdS_4 \times S^7$ found in [40]. It turns out that part of them are particular solutions of a Neumann-Rosochatius system. Let us explain this in more detail. Consider the membrane embedding (3.20), which in view of (3.22), reduces the Lagrangian (3.21) to (3.23). By choosing

$$\omega_{12} = \omega_{22} = \omega_{31} = \omega_{41} = 0$$

[9] For the limits of the integrals over ξ see [9].

as in (3.13), one obtains

$$\sum_{a<b=1}^{4} (\omega_{a1}\omega_{b2} - \omega_{a2}\omega_{b1})^2 r_a^2 r_b^2 = (\omega_{11}^2 r_1^2 + \omega_{21}^2 r_2^2)(\omega_{32}^2 r_3^2 + \omega_{42}^2 r_4^2).$$

For $r_{3,4} = constants$, $C_3 = C_4 = 0$, this leads to a Neumann-Rosochatius Lagrangian of the type (3.24)

$$L = \frac{(4l_p\mathcal{R})^2}{4\lambda^0} \sum_{a=1}^{2} \left[\dot{r}_a^2 - \left(8\lambda^0 T_2 l_p \mathcal{R}\omega_r\right)^2 \omega_{a1}^2 r_a^2 - \frac{C_a^2}{r_a^2}\right] + \Lambda_S \left[\sum_{a=1}^{2} r_a^2 - (1 - r_3^2 - r_4^2)\right],$$

where

$$\omega_r^2 = \omega_{32}^2 r_3^2 + \omega_{42}^2 r_4^2.$$

Now, let us impose the conditions

$$\alpha_a(\tau) = \omega_{a0}\tau, \quad \omega_{a0} = constants,$$

as is the case in [40]. These are compatible with (3.22) for $r_a = constants$ only, i.e. for the constant radii solutions of [40], when the equations of motion and constraints reduce to relations between the free parameters of the membrane embedding. Therefore, the membrane solutions described in section 4 of [40] are solutions of a Neumann-Rosochatius system for particular choice of the parameters ω_{ai} and C_a. For ω_{ai} - arbitrary, they are solutions of a more general system, given by the Lagrangian (3.23).

According to AdS-CFT correspondence, strings on $AdS_5 \times S^5$ and membranes on $AdS_4 \times S^7$ are dual to *different* gauge theories. Hence, one is tempting to conjecture that there should exist common integrable sectors on the field theory side.

We expect that in the framework of our approach, one can find relations between membranes in $AdS_7 \times S^4$ and Neumann and Neumann-Rosochatius like integrable systems with indefinite signature, analogous to (2.13) and (2.19).

On the other hand, we observed that only a small class of membrane configurations described by the embedding (3.8) are captured by the Neumann and Neumann-Rosochatius dynamical systems. Actually, these configurations are *exceptional*, taking into account the Lagrangians (3.12), (3.17), (3.21), (3.26) and (3.30). The conclusion is that there exist many possibilities for discovering, known or new, integrable systems dual to the membranes in M-theory.

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